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BAER ORDERED $*$-FIELDS OF THE FIRST KIND

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ABSTRACT

We construct examples of Baer ordered $*$-fields of the first kind of every dimension 4^n, $n = 1, 2, \ldots$.

1. Introduction. Lifting a Baer ordering

This paper presents examples of Baer ordered $*$-fields supplementing the examples contained in [6], [7], and [8]. I use the term "$*$-field" as a convenient term for what is often called a "division ring with involution." We shall assume characteristic $\neq 2$ throughout. By *dimension* of a field K, I mean the dimension of K as a vector space over its center.

We say that a $*$-field K with involution $\alpha \to \alpha^*$ has a *Baer ordering* if there exists a subset Π of K (called the *domain of positivity* for the ordering) satisfying these five conditions: (1) Π consists solely of symmetric elements ($\alpha = \alpha^*$); (2) contains 1 but not 0; (3) is closed under sum; (4) contains along with λ every $\alpha\lambda\alpha^*$, $\alpha \neq 0$; and (5) contains either λ or $-\lambda$ (but not both) for every nonzero symmetric λ. The relation $\lambda < \mu \Leftrightarrow \mu - \lambda \in \Pi$ then totally orders the set of symmetric elements. This definition was proposed by Reinhold Baer in his 1952 book [4, Chapter IV, Appendix I].

When the center of a $*$-field consists solely of symmetric elements, we call the involution $*$ *of the first kind.* If the center contains a nonsymmetric element, in which case the center is a quadratic extension of its subfield of symmetric elements, we call the involution *of the second kind*, or *unitary.* A noncommutative finite-dimensional $*$-field of the first kind has dimension 4^n, $n = 1, 2, \ldots$ [1; Thm. X.19 and Thm. V.17].

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In [7] I have constructed examples of finite-dimensional Baer ordered $*$-fields of the second kind of dimension p^2, for every prime $p \equiv 3 \pmod 4$. The following theorem constitutes the main result of this paper.

1.1. THEOREM. *There exist Baer ordered $*$-fields of the first kind of every dimension* 4^n, $n = 1, 2, \ldots$.

The examples are tensor products of quaternion fields. We show them ordered by the "lifting" procedure described in [7]. Here is a brief description of the procedure.

1.2. LEMMA. *Let K and K_0 denote $*$-fields, and let N: $K \to K_0$ be a mapping of K onto K_0 that satisfies the following four conditions*:

(1) $N(\alpha) = 0 \Leftrightarrow \alpha = 0$; $N(1) = 1$,

(2) *for each α and each symmetric λ in K, there exists β in K_0 such that* $N(\alpha\lambda\alpha^*) = \beta N(\lambda)\beta^*$,

(3) *if α is symmetric, so is $N(\alpha)$,*

(4) *if $N(\alpha) + N(\beta) \neq 0$, then $N(\alpha + \beta) =$ one of $N(\alpha)$, $N(\beta)$, $N(\alpha) + N(\beta)$. Suppose K_0 is Baer-ordered with domain of positivity Π_0. Then the set $\Pi = \{\lambda \in K;\ \lambda = \lambda^*$ and $N(\lambda) \in \Pi_0\}$ is a domain of positivity in K.*

We shall refer to this procedure as "lifting the ordering" from K_0 to K. To prove Lemma 1.2, we just check that our Π satisfies the five conditions listed at the beginning of this section. By definition, Π consists solely of symmetric elements, and we have $1 \in \Pi$ because $N(1) = 1 \in \Pi_0$. Also $0 \notin \Pi$ because $N(0) = 0 \notin \Pi_0$. Thus our Π satisfies the first two requirements. Next we must show that if λ, $\mu \in \Pi$, then also $\lambda + \mu \in \Pi$. This amounts to proving $N(\lambda) \in \Pi_0$, $N(\mu) \in \Pi_0 \Rightarrow N(\lambda + \mu) \in \Pi_0$. But if $N(\lambda) \in \Pi_0$ and $N(\mu) \in \Pi_0$, then $N(\lambda) + N(\mu) \in \Pi_0$ so $N(\lambda) + N(\mu) \neq 0$, hence (by (4) of Lemma 1.2)), $N(\lambda + \mu) =$ one of $N(\lambda)$, $N(\mu)$, $N(\lambda) + N(\mu)$, all of which lie in Π_0. Thus Π is closed under sum, which is the third requirement. If $\lambda \in \Pi$, which means $N(\lambda) \in \Pi_0$, and $0 \neq \alpha \in K$, then by hypotheses (2) and (1), $N(\alpha\lambda\alpha^*) = \beta N(\lambda)\beta^*$ for a nonzero β in K_0, hence $\alpha\lambda\alpha^* \in \Pi$.

To establish the last requirement on Π, we need this remark: *A function N that has the four properties listed in Lemma* 1.2, *also has the property $N(-\alpha) = -N(\alpha)$ for all $\alpha \in K$.* To prove this remark we may assume $\alpha \neq 0$ (as the remark is obviously true when $\alpha = 0$). We need to show that $N(-\alpha) + N(\alpha) = 0$. Suppose not. Then, by (4), $N(-\alpha + \alpha) =$ one of $N(-\alpha)$, $N(\alpha)$, $N(-\alpha) + N(\alpha)$. But none of these is 0, while $N(-\alpha + \alpha) = N(0) = 0$. Hence we must have $N(-\alpha) + N(\alpha) = 0$.

Returning to the fifth requirement on Π, suppose that $0 \neq \lambda = \lambda^* \in K$. Then by (3) and (1), $N(\lambda)$ is a nonzero symmetric in Π_0, so either $N(\lambda) \in \Pi_0$, or $-N(\lambda) = N(-\lambda) \in \Pi_0$. In the first case, $\lambda \in \Pi$, and in the second case, $-\lambda \in \Pi$. That proves Lemma 1.2.

In the applications of this lemma, K is the $*$-field we wish to order, K_0 is its residue class $*$-field with respect to a $*$-valuation, and the map N is constructed with the help of what is called a smooth presection for the $*$-valuation. The details of the construction of N are, briefly, as follows. (This material is taken from §§2 and 3 of [7].)

A $*$-valuation on a $*$-field K is a homomorphism w of $K^\times$, the multiplicative group of nonzero elements of K, onto a (necessarily abelian) ordered group G such that $w(\alpha^*) = w(\alpha)$ for all $\alpha \in K^\times$, and $w(\alpha + \beta) \geqq \min\{w(\alpha), w(\beta)\}$ for all $\alpha, \beta \in K$ with $\alpha + \beta \neq 0$.

Given a $*$-valuation $w\colon K^\times \to G$, the set $\Phi = \{\alpha \in K^\times\colon w(\alpha) \geqq 0\} \cup \{0\}$ is the *$*$-valuation ring* of w, $\mathcal{R} = \{\alpha \in K^\times\colon w(\alpha) > 0\} \cup \{0\}$ the maximal ideal of Φ, and $K_0 = \Phi/\mathcal{R}$ the *residue class $*$-field* of w, which carries the natural induced involution defined by $(\alpha + \mathcal{R})^* = \alpha^* + \mathcal{R}$, $\alpha \in \Phi$. We use θ for the natural map of Φ onto K_0: $\theta(\alpha) = \alpha + \mathcal{R}$. Hence, by definition, $\theta(\alpha^*) = \theta(\alpha)^*$. Given a symmetric or skew element ρ in $K^\times$, the formula $\alpha^\# = \rho\alpha^*\rho^{-1}$ defines another involution on K which in turn induces another involution on K_0 which is effectively defined by $\theta(\alpha)^\# = \theta(\rho\alpha^*\rho^{-1}) = \theta(\alpha^\#)$, $\alpha \in \Phi$. I call the element ρ *smooth* if this induced involution on K_0 is conjugate to its $*$. That is, ρ is smooth when there exists an automorphism Γ_ρ of K_0 so that

$$\# = \Gamma_\rho \circ * \circ \Gamma_\rho^{-1} \qquad \text{on } K_0.$$

I call the $*$-valuation w itself *smooth* when it fulfills the following two conditions:

(1) $w(2) = 0$;

(2) each equivalence class $w^{-1}(g)$ contains a smooth symmetric element if it contains symmetric elements at all, otherwise it contains a smooth skew element.

On the basis of these definitions, one can state the main theorem concerning the lifting of orderings as follows.

1.3. THEOREM [7; Theorem 3.2]. *Given a $*$-field K with smooth $*$-valuation, then there exists a map N of K onto its residue class $*$-field K_0 satisfying the conditions listed in Lemma 1.2. Hence any Baer ordering of K_0 may be lifted to K.*

We refer to [7] for the detailed proof of Theorem 1.3, and continue now our sketch of the construction of the map N.

The next step is to show that a smooth $*$-valuation $w\colon K^\times \to G$ always has a *smooth presection* $s\colon G \to K^\times$ which is a map that selects, for each $g \in G$, an $s(g) \in w^{-1}(g)$ together with a corresponding automorphism $\Gamma_{s(g)}$ of the residue class $*$-field K_0, such that the involution $x^\# = s(g)x^*s(g)^{-1}$ on K induces the involution $\Gamma_{s(g)} \circ * \circ \Gamma_{s(g)}^{-1}$ on K_0, and such that the following four conditions are fulfilled.

(1) $s(0) = 1$ and $\Gamma_1 = I$,

(2) $s(g)$ is smooth symmetric if $w^{-1}(g)$ contains symmetric elements, otherwise smooth skew,

(3) given $g = 2h$ in $2G$, there exist $\beta \in K^\times$ such that $s(2g) = \beta\beta^*$ and $\Gamma_{s(2g)}(x) = \beta x \beta^{-1}$ on K_0,

(4) given $g \in G$ written (not necessarily uniquely) $g = h + 2k$, then there exists $\gamma \in K^\times$ such that $s(g) = \gamma s(h)\gamma^*$ and $\Gamma_{s(g)}(x) = \gamma\Gamma_{s(h)}(x)\gamma^{-1}$ on K_0.

That construction of the smooth presection s is the key step. With it in hand, the function N is defined thus: if $0 \neq \alpha \in K^\times$ with $w(\alpha) = g$, set $N(\alpha) = \Gamma_{s(g)}^{-1} \circ \theta(\alpha s(g)^{-1})$, where, as above, θ is the natural map of the $*$-valuation ring Φ onto K_0, and $s\colon G \to K^\times$ is the smooth presection. Complete the definition of N by setting $N(0) = 0$. Thus defined, this function N, which maps K onto its residue class $*$-field K_0, fufills the requirements of Lemma 1.2. The paper [7] provides details of the proof, and §4 of [8] some additional remarks. A more informal and more detailed proof of the key lemma that any smooth $*$-valuation has a smooth presection is available in preprint form.

2. The examples

We begin with a general discussion of $*$-fields of the first kind. The substance of this exposition is taken mainly from §5 of L. Rowen's paper [12] to which we refer for proofs and other references.

If K is a $*$-field with involution $\alpha \to \alpha^*$, and ρ a nonzero element in K that satisfies $\rho = \rho^*$, then the formula $x^\# = \rho x^* \rho^{-1}$ defines another involution $\#$ on K that Rowen calls *equivalent* to $*$. This is an equivalence relation among involutions on K. If $\rho^* = -\rho$, then the same formula $x^\# = \rho x^* \rho^{-1}$ also defines another involution $\#$ on K that we call *skew-equivalent* to $*$. If $*$ is unitary, then equivalence and skew-equivalence coincide. But if $*$ is of the first kind, then equivalent involutions are never skew-equivalent, and skew-equivalent involutions never equivalent. Moreover, in this case, skew-equivalence is not an equivalence relation.

(In these terms we can rephrase the "smoothness" criterion mentioned in §1: A symmetric (resp. skew) element ρ is smooth for a $*$-valuation if the equivalent (resp. skew-equivalent) associated involution $x^{\#} = \rho x^* \rho^{-1}$ induces on the residue class *-field an involution conjugate to its *.)

If K is a *-field of the first kind of dimension 4^n over its center Z, then the symmetric elements within K form a vector space over Z of dimension either $\frac{1}{2}(4^n - 2^n)$ or $\frac{1}{2}(4^n + 2^n)$. In the former case one calls the involution *symplectic*; in the latter case, *orthogonal*. All symplectic involutions on K are mutually equivalent, all orthogonal involutions mutually equivalent, and any symplectic involution is skew-equivalent to any orthogonal one. Thus K has exactly two equivalence classes of first kind involutions, the symplectic and the orthogonal.

The fact that K has only two equivalence classes of involutions simplifies the discussion of the Baer orderability of K. The property "Baer orderable" *scales* [8; §2], which is to say it is preserved under equivalence. Because, if Π is a domain of positivity for $K(*)$, then $\Pi\rho^{-1}$ qualifies as a domain of positivity for $K(\#)$ where $x^{\#} = \rho x^* \rho^{-1}$, $\rho^* = \rho \in \Pi$. (We may take $\rho \in \Pi$ as replacing ρ by $-\rho$ does not change $x^{\#}$.) Hence if a *-field is Baer orderable for its given involution, then it is also Baer orderable for any equivalent involution. Thus, *in discussing the Baer orderability of a finite-dimensional *-field of the first kind, there are just two cases to consider: the symplectic and the orthogonal.*

We turn now to the construction of the example that verifies Theorem 1.1. This example is a special case of the iterated Laurent series division rings which Amitsur brought to prominence in his construction of a noncrossed product [2]. Amitsur constructed these division rings (which he used to show that certain generic matrix algebras could not be crossed products) by making successive Laurent series adjunctions of $2n$ indeterminates $x_1, y_1, \dots, x_n, y_n$ to a commutative base field F, all indeterminates commuting except that $y_i x_i = \omega_i x_i y_i$, ω_i a q_i root of unity, q_i prime, $1 \leqq i \leqq n$. One may find a detailed description in [9; II.5], and an in-depth study in a more general setting in [13]. Our example, which we shall present now in detail, has all $q_i = 2$, so all $\omega_i = -1$.

Let F be a commutative field, q-ordered in the sense of Prestel [11]. Any ordinary commutative ordered field is q-ordered, so we may take any such field for F, for example the rationals $\mathbf{Q}$ or the real numbers $\mathbf{R}$. Let $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ be $2n$ indeterminates over F with the multiplication table

$$(1)\qquad \begin{aligned} &x_i\zeta = \zeta x_i \qquad y_i\zeta = \zeta y_i \quad \text{for } \zeta \in F\\ &x_i x_j = x_j x_i \qquad y_i y_j = y_j y_i\\ &y_i x_j = x_j y_i \quad \text{when } i \neq j\\ &y_i x_i = -x_i y_i. \end{aligned}$$

Let $K = F((x_1, y_1, \dots, x_n, y_n))$ be the iterated Laurent series field obtained by successively adjoining the indeterminates $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ to F in the order indicated. The first adjunction gives the commutative field $F((x_1))$ whose general nonzero term has the form $\phi(x_1) = \Sigma \zeta_i x_1^i$ where $\zeta_i \in F$ and the sum is taken over a subset of the integers that has a smallest element $a(1)$ with corresponding coefficient $\zeta_{a(1)} \neq 0$. At the next stage we get the field $F((x_1, y_1))$ whose general nonzero term has the form $\phi(x_1, y_1) = \Sigma \phi_i(x_1) y_1^i$ where $\phi_i(x_1) \in F((x_1))$ and the sum is taken over a subset of the integers that has a smallest element $b(1)$ with corresponding coefficient $\phi_{b(1)}(x_1) \neq 0$. Add term-by-term and multiply according to the rules in table (1). At the last stage (which yields K) we adjoin y_n to $F((x_1, y_1, \dots, x_n))$. The general nonzero element α of K has the form $\alpha = \Sigma \phi_i(x_1, y_1, \dots, x_n) y_n^i$ where $\phi_i \in F((x_1, y_1, \dots, x_n))$ and the sum is taken over a subset of the integers that has a smallest element $b(n)$ with corresponding coefficient $\phi_{b(n)}(x_1, y_1, \dots, x_n) \neq 0$.

Let $\mathbf{Z}$ denote the integers, and $\mathbf{Z} \times \cdots \times \mathbf{Z} = \mathbf{Z}^{2n}$ the abelian group of $2n$-tuples of integers with the antilexicographic ordering, i.e. the ordering whose positive elements are those $2n$-tuples $(m_1, \dots, m_{2n})$ with $m_i > 0$, $m_j = 0$, $j > i$. In the expression for the general nonzero element α of K let $b(n)$ denote the smallest power of y_n that appears in $\alpha = \Sigma \phi_i(x_1, y_1, \dots, x_n) y_n^i$ with nonzero coefficient $\phi_{b(n)}(x_1, y_1, \dots, x_n)$, let $a(n)$ denote the smallest power of x_n that appears in $\phi_{b(n)}(x_1, y_1, \dots, x_n) = \Sigma \phi_i(x_1, \dots, y_{n-1}) x_n^i$ with nonzero coefficient $\phi_{a(n)}(x_1, \dots, y_{n-1})$, etc., just as before. We may then write the general nonzero α in K as

$$\alpha = \zeta x_1^{a(1)} y_1^{b(1)} \cdots x_n^{a(n)} y_n^{b(n)} + \cdots \tag{2}$$

where the ellipsis represents a sum over $2n$-tuples strictly greater than $(a(1), \dots, b(n))$ in the antilexicographic ordering, and $0 \neq \zeta \in F$.

With α expressed as in (2), define

$$w(\alpha) = (a(1), b(1), \dots, a(n), b(n)) \in \mathbf{Z}^{2n}. \tag{3}$$

One checks easily that w is a Krull valuation on K, mapping $K^\times$, the multiplicative group of nonzero elements of K, onto the totally ordered abelian group $\mathbf{Z}^{2n}$. The residue class field of w is clearly F.

The center Z of K is the commutative iterated Laurent series field

$$Z = F((x_1^2, y_1^2, \dots, x_n^2, y_n^2)),$$

and K is obtained from Z by the algebraic adjunction of the quadratic elements $x_1, y_1, \dots, x_n, y_n$ to Z. If K_i stands for the quaternion field obtained by adjoining

x_i and y_i to Z, then we may represent our field K as the tensor product

$$K = \bigotimes_{i=1}^{n} K_i,$$

the tensor product being taken over the common center Z. In this form the field K was first constructed by Köthe [10; §3]; one may find a description in [5; Ch. II].

The quaternion field K_i is spanned over Z by the four elements 1, x_i, y_i, x_iy_i. A quaternion field admits exactly one symplectic first kind involution; for K_i we get this involution by making x_i and y_i skew. (This choice makes x_iy_i also skew.) A quaternion field also admits many (mutually equivalent) orthogonal first kind involutions. For K_i, we get such an involution by making x_i and y_i both symmetric (which makes x_iy_i skew). Any choice of first kind involutions on each of the n fields K_i induces a first kind involution on K [1; Ch.X, Lemma 2] which is either symplectic or orthogonal. The involution induced on K will be symplectic when we have an odd number of symplectic factors K_i; otherwise it will be orthogonal. We shall prove the Baer orderability in either case, thus proving more than is explicitly stated in Theorem 1.1.

Let $*$ be the symplectic or orthogonal involution induced on K, in the manner just referred to, by various assigned first kind involutions on the K_i. If the nonzero element α of K has the representation given as in (2)

$$\alpha = \zeta x_1^{a(1)} y_1^{b(1)} \cdots x_n^{a(n)} y_n^{b(n)} + \cdots$$

then

$$\alpha^* = \pm \zeta x_1^{a(1)} y_1^{b(1)} \cdots x_n^{a(n)} y_n^{b(n)} + \cdots .$$

Hence $w(\alpha) = w(\alpha^*)$, so the valuation w defined by (3) is a $*$-valuation.

The valuation w is also smooth (in the terminology of Section 1), in fact with the associated automorphisms Γ_ρ of the residue class $*$-field F always the identity. This is a consequence of the fact that $\theta(\beta\alpha\beta^{-1}) = \theta(\alpha)$ for all α in Φ, and all β in $K^\times$. Here $\Phi = \{\alpha \in K^\times; w(\alpha) \geqq 0\} \cup \{0\}$ is the $*$-valuation ring of w, and θ is the natural map of Φ onto the residue class $*$-field $\Phi/\mathcal{R}$, where $\mathcal{R} = \{\alpha \in K^\times; w(\alpha) > 0\} \cup \{0\}$. Using the notation (2), the general nonzero element α of Φ can be written

$$\alpha = \zeta + \cdots$$

where $\zeta \in F$. We have $\zeta \neq 0 \Leftrightarrow \alpha \in \Phi^\times = \{\alpha \in K^\times; w(\alpha) = 0\}$ the group of invertible elements in Φ. If $\alpha \in \Phi$, the map θ is given by $\theta(\alpha) = \zeta$, and we have clearly

$$\beta\alpha\beta^{-1} = \zeta + \cdots$$

for every $\beta \in K^\times$, thus $\theta(\beta\alpha\beta^{-1}) = \theta(\alpha)$ for every $\alpha \in \Phi$, and every β in $K^\times$. Referring back to the discussion directly preceding Theorem 1.3, we see that accordingly every nonzero symmetric and skew element ρ is smooth, with associated automorphism $\Gamma_\rho =$ identity. As each equivalence class $w^{-1}(g)$ contains either a symmetric or a skew element [7; p. 224], our $*$-valuation w is smooth. Thus, by Theorem 1.3, we can lift the Baer ordering from F to K (Prestel's q-ordering, which is the ordering we have put on F, is the special case of Baer's ordering with $*$-identity).

Our method of proof is constructive, and permits us to explicitly describe the lifted ordering, once having constructed a smooth presection. For example, consider one quaternion $*$-field K spanned by x, y, and xy. Put an orthogonal involution on K by making x and y symmetric. The general nonzero element α of K has the form

$$\alpha = \phi_n(x)y^n + \phi_{n+1}(x)y^{n+1} + \cdots = \zeta x^m y^n + \cdots$$

where $\phi_n(x) = \zeta x^m + \cdots$, $0 \neq \zeta \in F$. The function $N(\alpha) = \theta(\alpha s(g)^{-1})$ gives us an explicit rule to determine which symmetric α are positive and which are negative.

To construct the smooth presection s, first define an auxiliary function $t(p,q) = x^p y^q$ which selects for each $g = (p,q) \in G = \mathbf{Z} \times \mathbf{Z}$ an α in K with $w(\alpha) = g$. Next define the presection s on $2G$ by $s(2p,2q) = t(p,q)t(p,q)^*$ $= x^{2p}y^{2q}$. The elements $x^{2p}y^{2q}$ are central symmetrics.

We select $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$ as representatives of the four cosets of $2G$ in G, define s on these elements as 1, x, y, and xy (skew) respectively, and the define s on all of G by $s(g) = t(h)s(a)t(h)^*$ where $g = a + 2h$ (unique), a being one of the coset representatives. We find $s(m,n) = (-1)^{\varepsilon(m,n)}x^m y^n$ where $\varepsilon(2p,2q) = 0$, $\varepsilon(2p+1,2q) = q$, $\varepsilon(2p,2q+1) = p$, and $\varepsilon(2p+1,2q+1) = p+q$. The function $N: K \to F$ then has this explicit form: $N(\zeta x^m y^n + \cdots) = (-1)^{\varepsilon(m,n)}\zeta$, $0 \neq \zeta \in F$. Hence $N(x) = N(y) = 1$ so x and y are positive. But $N(x^2y) = N(xy^2) = -1$ so the symmetrics x^2y and xy^2 are both negative. In a q-ordered commutative field, any nonzero square times a positive element is positive. As $x^2y < 0$, the same rule does not hold in a Baer ordered $*$-field, even though x^2 and y commute. But $x^2y < 0$ is also clear on other grounds because $y > 0 \Rightarrow 0 < xyx^* = xyx = -x^2y$. Also $x^2y^2 > 0$, $x^6y < 0$, etc.

Further $1 - nx = 1 \cdot x^0y^0 - nx^1y^0 > 0$ for all $n = 1, 2, \ldots$, so $x < (1/n)$ for all n, thus x is infinitesimal. And $x - ny = x^1y^0 - nx^0y^1 > 0$ for all n, so $y < (1/n)x$, $n = 1, 2, \dots$.

Clearly the same kind of explicit description can be given for any number of quaternion factors.

That completes the proof of Theorem 1.1.

On page 227 of [7] I made the statement: "Surprisingly, the tensor product of quaternion $*$-fields with its usual involution never admits an ordering," which I corrected in the erratum to include the additional hypothesis that a basis element of one of the components has square congruent to -1 modulo the common center. In the erratum I also asserted: "Whether the nonorderability continues to hold without the qualifying restriction on a basis element seems to be an open question."

Clearly the question is no longer open, as the examples of Baer-ordered $*$-fields that we have just constructed are all tensor products of quaternion $*$-fields.

Moreover, the qualified statement, which was the one actually proved in [7], is a special case of the following interesting result told to me by Maurice Chacron.

2.1. THEOREM (Chacron). *Suppose K is a noncommutative Baer-ordered $*$-field with center Z, and suppose further that K is not a standard quaternion $*$-field. Then, given $0 \neq \zeta \in Z$, the equation $x^2 = -\zeta^2$ has only centeral skew solutions x. In fact, $x^2 = -\zeta^2$, $0 \neq \zeta \in Z$, has no solutions at all when K is of the first kind, and has a solution if and only if $\sqrt{-1} \in Z$ and $(\sqrt{-1})^* = -\sqrt{-1}$ when K is of the second kind. In the latter case, $x = \zeta\sqrt{-1}$ is the unique solution.*

By a "standard quaternion $*$-field" I mean a 4-dimensional field equipped with its unique symplectic involution. (In this case the center consists exactly of the symmetric elements.)

PROOF (M. Chacron, private communication). By considering x/ζ in place of x, we may clearly deal with the equation $x^2 = -1$.

The proof uses the fact that a Baer-ordered $*$-field is formally real, which means that an equation $\Sigma\alpha_i\sigma\alpha_i^* = 0$, where $\sigma = \sigma^*$, can have only a trivial solution.

First, note that if $x^2 = -1$ has a solution at all, then x must be skew. Write $x = \sigma + \tau$, $\sigma^* = \sigma$, $\tau^* = -\tau$. Then $x^2 = \sigma^2 + \tau^2 + (\sigma\tau + \tau\sigma)$, so $\sigma^2 + \tau^2 = -1$, $\sigma\tau + \tau\sigma = 0$. A routine calculation shows then that $\tau\sigma\tau^* + \sigma + \sigma\sigma\sigma^* = 0$ whence, by formal reality, $\sigma = 0$. Hence $x = \tau$ is skew.

If $0 \neq \rho = \rho^*$, then $\lambda = x\rho - \rho x$ is symmetric, and by direct computation

$x\lambda x^* + \lambda = 0$. Using formal reality again, we conclude that $\lambda = 0$. Hence x commutes with every symmetric element, thus must be central by Dieudonné's lemma [6; Lemma 1], as we have assumed K is noncommutative and non-quaternionic. The remaining assertions in Theorem 2.1 now follow routinely.

Chacron's theorem generalizes the result stated in the erratum of [7]. It also shows that the example constructed by Amitsur, Rowen, and Tignol [3] of a $*$-field of the first kind of dimension 4^3 not the tensor product of quaternions cannot be Baer ordered, because this first-kind $*$-field contains an element ξ_1 satisfying $\xi_1^2 = -1$ [3, Theorem 5.1, first line of proof].

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